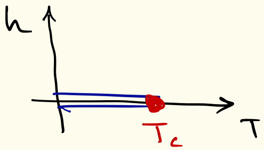
In $d > 1$, Ising model shows a order-disorder transition at $T = T_c > 0$, and as we discussed in 210A, the phase diagram looks like

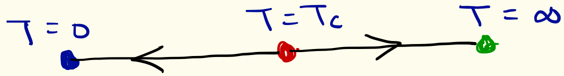


In $d = 2$, the Ising model is exactly solvable and one can map it to a system of free fermions using Jordan-Wigner transformation. We will skip this discussion since our focus is RG. We simply note that the critical exponent β corresponding to $m \sim (T_c - T)^\beta$ is $\beta = \frac{1}{8}$, and the exponent ν corresponding to $\xi \sim |T - T_c|^{-\nu}$ is $\nu = 1$.

These exponents are very different from the mean-field ones, which was historically a prime motivation behind RG.

In $d \geq 4$, the Ising critical exponent is essentially mean-field like and the exponents $\beta = \frac{1}{2}$, $\nu = \frac{1}{2}$.

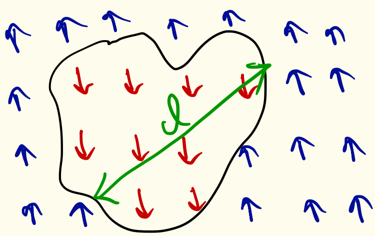
Based on the above discussion, the RG flow for Ising model in $d>1$ is expected to look like:



That is, the ordered phase is stable to thermal fluctuations. Let's recall the domain wall argument from 210A and then perform a real-space RG calculation, which will also highlight the difficulties associated with performing real-space RG in a systematic manner.

Domain-Wall argument in $d=2$:

The $T=0$ order will be destroyed if and only if there are domain walls of arbitrary large size i.e. if the typical size of a domain wall scales with the total system size. Therefore consider a domain wall of size $l \propto L$ (= total linear system size)



Energy cost of the domain wall $\sim J \times$ length of the domain wall

$$= J\, l$$

Entropy cost $\sim \log(\#$ of SAWs of l steps$)$

$\sim l$

Free energy contribution $\sim J l - T l$

$= (J-T) l$

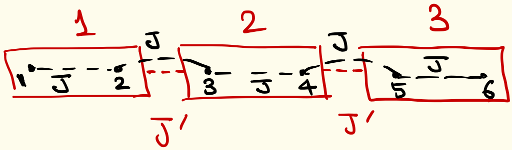
Thus, as $l \sim L \to \infty$, large domain walls are suppressed below $T \sim J$. This suggests an order-disorder transition at $T_c = O(J)$.

What about small domain walls, e.g. single spin flips. These are point-like particles and there will be a finite density of them at any non-zero T: density $n \sim e^{-\beta J}$. The effect on the magnetization of these tiny domain walls is however also tiny: $\Delta m \sim \frac{L^2 - L^2 n}{L^2}$

$\sim 1 - n \sim 1 - e^{-\beta J} \sim 1$ at $T \ll J$.

RG for 2d Ising model:

In 1d, at low T, the RG flow to $T(\ell)$ at the leading order was $dT/d\ell = 0$ (recall T was only marginally relevant). The reason for this is that the effective interaction between block spins is proportional to the # of original spins mediating the interaction:



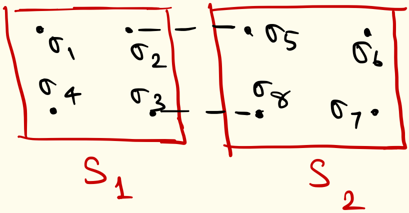
Denoting the original spins as σ, and coarse-grained (block) spins as S,

$$J' \sim J \langle \sigma_2 \rangle_{S_1 = 1} \langle \sigma_3 \rangle_{S_3 = 1}$$

where $\langle \sigma_i \rangle_{S_k = 1}$ is the value of $\langle \sigma_i \rangle$ conditioned on $S_k = 1$. At low T, this is just unity (otherwise S_k won't equal 1).

$\Rightarrow$ $J' \sim J$, or equivalently $T' \sim T$ at the leading order.

Now consider the same argument in $d=2$



$$J' \sim J\left[\langle \sigma_2 \sigma_5 \rangle + \langle \sigma_3 \sigma_8 \rangle\right]$$

$$\sim 2J$$

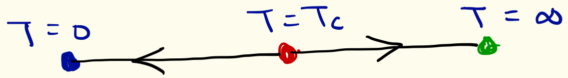
Thus, the coarse-grained effective coupling is larger than the original coupling. For a general coarse-graining factor of b in d-dim (above, $b=2$): $J' = b^{d-1} J$, or at the leading order RG flow is

$$\frac{dJ}{dl} = (d-1)J + \ldots$$

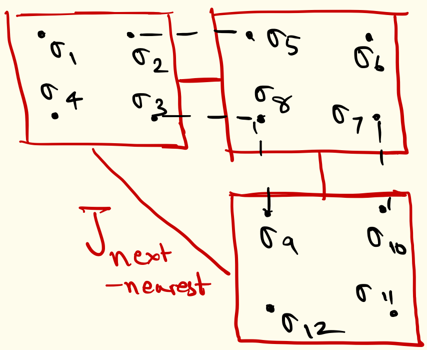
or, in terms of temperature T $(\sim J^{-1})$

$$\frac{dT}{dl} = (1-d)T + \ldots$$

This implies that the $T=0$ ordered, spontaneous sym. broken phase, is stable at small temperatures. Since the $T=\infty$ paramagnet fixed point is always stable, the simplest possibility is the existence of an intermediate fixed point, that corresponds to the order-disorder critical point:



Before we perform a more detailed RG that substantiates this result, let's note a complication that arises in $d>1$, which is absent in $d=1$: even if the original Hamiltonian has only nearest-neighbor interactions, the coarse-grained H will have interactions that are not:

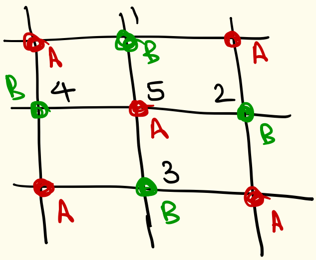


In general, the effective Hamiltonian will contain all terms that are allowed by the

Ising sym. This severely complicates real-space RG schemes and one has to typically resort to some sort of truncation scheme.

Let's consider a few different real space schemes for $d > 1$ Ising model:

(1) Integrating out a subset of spins:



Consider square lattice ising model, which we write as

$$H = -J \sum_{\langle r r' \rangle} S_A(r)\, S_B(r')$$

where A and B denotes the two sublattices (denoted by red and green vertices above). Similar to the RG for 1d Ising model, one may trace out one of the sublattices (say the A sublattice) to obtain an effective Hamiltonian for the untraced sites (= B sublattice). Let's do this shematically.

$$\mathrm{Tr}\; e^{J s_5 \left(\sum_{i=1}^{4} S_i \right)} = e^{\log 2\cosh\left(J \sum_{i=1}^{4} S_i\right)}$$

$$= C\, e^{J' \sum_{ij} S_i S_j + K S_1 S_2 S_3 S_4}$$

where we have used the Ising symmetry and the pertumation sym between s_1 to s_4 to write the schematic expression. We make a few observations at this point :

(i) The renormalized H is defined on a square lattice tilted by $\frac{\pi}{4}$ w.r.t. to the original one and has a lattice spacing that is $\sqrt{2}$ times larger.

(ii) Although the renormalized Hamiltonian has the symmetry (Ising) as the original one, it is longer ranged (e.g. involves diagonal interactions between sites 1 and 3), and also unconventional interactions such as $s_1 s_2 s_3 s_4$. This is a general feature of RG : all terms allowed by sym. will be generated.

Since J' is the analog of renormalized J, we would be most interested in its value.

$$2\cosh\left(J\sum_{i=1}^{4} s_i\right) = c\, e^{J'\sum_{ij} s_i s_j + K s_1 s_2 s_3 s_4}$$

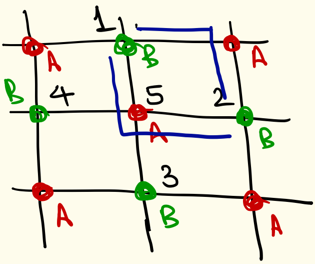
$s_i = 1 \;\forall i \quad \Rightarrow \quad 2\cosh(4J) = c\, e^{6J' + K}$

$s_1 = s_2 = s_3 = 1,\ s_4 = -1 \Rightarrow \quad 2\cosh(2J) = c\, e^{-K}$

$s_1 = s_2 = 1,\ s_3 = s_4 = -1 \Rightarrow \quad 2 = c\, e^{-2J' + K}$

Instead of solving these equations exactly, lets consider a perturbative approach, where we keep only keep track of the nearest-neighbor coupling J_1 and next-nearest neighbor coupling J_2. Further, lets keep terms only to $O(J_1^2)$ and $O(J_2)$.

Renormalization of J_1:



The coupling between S_1 and S_4 gets contribution from

(i) original next-nearest coupling J_2 between them.

(ii) Two $S_A S_B S_A S_B$ pathways shown in the figure above. This contributes $2J_1^2$ since it's $O(J_1^2)$ and there are two pathways.

$$\text{Thus,} \quad J'_1 = J_2 + 2J_1^2.$$

Renormalization of J_2:

The original coupling between the next-nearest neighbors of the renormalized lattice (e.g. between S_1 and S_3) is zero. The pathway $J\, S_1 S_5 \; J\, S_5 S_3$ generates coupling $O(J_1^2)$.

$$\Rightarrow \quad J_2' = J_1^2$$

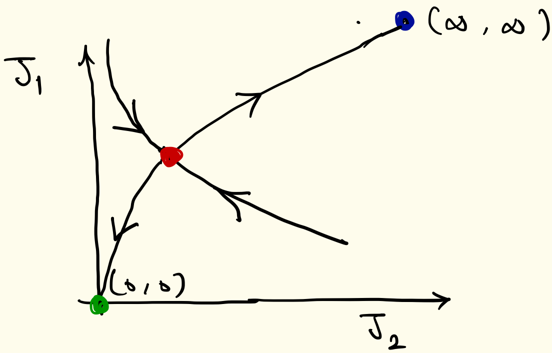
Solving the above coupled RG equations, one finds three fixed points:

$(J_1, J_2) = 0$: Infinite T paramagnet

$(J_1, J_2) = \infty$: $T = 0$ SSB fixed point.

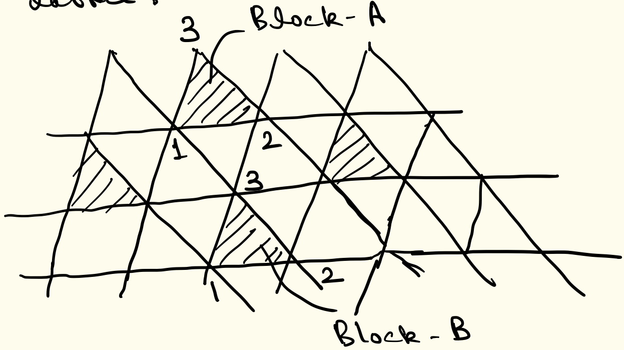
$(J_1, J_2) = (\frac{1}{3}, \frac{1}{9})$: Critical point between the ordered and disordered phases.

One may verify that the critical point has only one relevant direction, while the other two fixed points are stable.



(2) Majority-rule RG :

Let's consider a slightly different approach to real-space RG that is directly based on the majority rule blocking (↑↑↓ ~ ↑ etc.). This is based on sec 9.6 of Goldenfeld's book.. We will implement on a triangular lattice.



We define the effective Hamiltonian of the block spins as

$$e^{-H'(S')} = \mathrm{Tr}_S \, e^{-H(S)} \, T(S' | S)$$

where $T(S'|S)$ is the projection matrix that implements the majority rule:

S_1	S_2	S_3	S'
↑	↑	↑	↑
↑	↑	↓	↑
↑	↓	↑	↑
↓	↑	↑	↑

S' = coarse-grained block spin

Similarly for $S' = \downarrow$ configs.

The key idea is to separate the interaction between spins S of the original Hamiltonian into two groups : (i) intra-block (ii) interblock.

e.g. in the above figure, intra-block interactions

$$= -J \sum_{\alpha = A, B} [S^{\alpha}_1 S^{\alpha}_2 + S^{\alpha}_2 S^{\alpha}_3 + S^{\alpha}_3 S^{\alpha}_1]$$

while inter-block interactions $= -J(S^A_1 S^B_3 + S^A_2 S^B_3)$

This decomposition allows one to write

$$e^{-H'(s')} = \sum_{\{s\}} e^{-[H^{intra}(s) + H^{inter}(s)]} T(s'|s)$$

$$= \langle e^{-H_{inter}(s)} T(s'|s) \rangle_{intra} Z^{intra}$$

where $\langle A \rangle_{intra} = \dfrac{\mathrm{Tr}\ A\, e^{-H^{intra}}}{Z^{intra}}$

with $Z^{intra} = \mathrm{Tr}\ e^{-H^{intra}}$.

H^{intra} is Ising sym. $\Rightarrow$ Z^{intra} is independent of the coarse-grained variable S'.

Therefore we only need to take care of inter-block interactions, which we will implement perturbatively using cumulant expansion.

$$\left\langle e^{-H^{inter}(s)} \, T(s'|s) \right\rangle_{intra}$$

$$= e^{-\left[\langle H^{inter} \rangle'_{intra} - \frac{1}{2}\left[\langle (H^{inter})^2 \rangle'_{intra} - \langle H^{inter} \rangle'^{2}_{intra} \right] + \cdots \right]}$$

where the prime over averages ($\langle \; \rangle'$) denotes that the expectation value is projected onto the subspace consistant with the majority rule. As a concrete example, consider $\langle s_1^A \, s_3^B \rangle'_{intra}$. Since the intra-block Hamiltonians for blocks A and B are decoupled from each other,

$$\langle s_1^A \, s_3^B \rangle'_{intra} = \langle s_1^A \rangle'_{intra} \, \langle s_3^B \rangle'_{intra}$$

when $s'_A = +1$, $\langle s^A_1 \rangle'_{intra} = \dfrac{e^{3J} + e^{-J} + e^{-J} - e^{-J}}{e^{3J} + e^{-J} + e^{-J} + e^{-J}}$

$$= \frac{e^{3J} + e^{-J}}{e^{3J} + 3e^{-J}}$$

s_1	s_2	s_3	s'
↑	↑	↑	↑
↑	↑	↓	↑
↑	↓	↑	↑
↓	↑	↑	↑

On the other hand, when $S'_A = -1$, by Ising sym. $\langle S^A{}_1 \rangle'_{intra} = -\dfrac{(e^{3J}+e^{-J})}{e^{3J}+3e^{-J}}$

$$\Rightarrow \quad \langle S^A{}_1 \rangle'_{intra} = \frac{S'_A\,(e^{3J}+e^{-J})}{(e^{3J}+3e^{-J})}$$

$$\Rightarrow \quad \langle H^{inter} \rangle'_{intra} = -J \langle S^A{}_1 \rangle'_{intra} \langle S^B{}_3 \rangle'_{intra} - J \langle S^A{}_2 \rangle'_{intra} \langle S^B{}_3 \rangle_{intra}$$

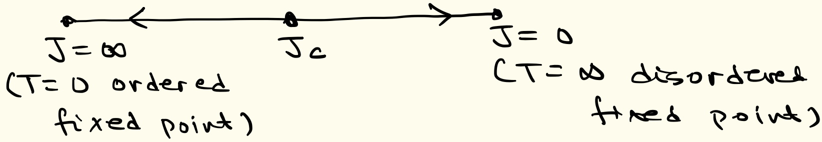
$$= -2J\left[\frac{e^{3J}+e^{-J}}{e^{3J}+3e^{-J}}\right]^2 S'_A\, S'_B$$

Therefore, at the leading order, the interaction between the renormalized spins is:

$$J' = J\left[\frac{e^{3J}+e^{-J}}{e^{3J}+3e^{-J}}\right]^2$$

Lets solve for the fixed points and critical exponents. Setting $J' = J$, one finds three fixed points: $J = 0$, $J = \infty$, and

$J_c = \frac{1}{4} \log(1 + 2\sqrt{2})$. One may check that the RG flow looks like



as expected for $d = 2$ Ising model.

The RG eigenvalue $\left.\frac{\partial J'}{\partial J}\right|_{J=J_c} = 1.62$

Since $b = \sqrt{3}$ $\Rightarrow$ $(\sqrt{3})^{y_t} = 1.62$

$\Rightarrow$ $y_t = 0.8782$.

The exact result is $y_t = 1$, so this is not too off.

By including contributions from higher-order cumulants, one can systematically improve this result.

Homework : Implement a similar RG scheme for the 1d Ising model.

(not to be submitted).